IJCEAS

*International Journal of Contemporary Economics and Administrative Sciences*
*ISSN: 1925 – 4423*
*Volume :5, Issue:3-4, Year:2015, pp.109 – 138*

# STRATEGIC DECOMPOSITION: HOW TO INCREASE TOTAL LABOR PRODUCTIVITY

Shu TSUCHIDA[1]

**Abstract**

*Labor productivity is a key indicator in economic analysis. Numerous aggregative formulae exist for calculating labor productivity growth at the firm-level, with the result that the contribution of each firm to total labor productivity growth varies according to the formulae used in its calculation. In this study, we systematically derive many aggregative formulae. Using one of these formulae, we examine the link between labor productivity growth at the firm-level and that at the industry-level, and identify the situations in which the former is not correlated with the latter. A central concept in this study is the clarification of a strategic decomposition. Our strategic decomposition uses a very simple formula and can be used to solve the problem of maximizing the increase of total labor productivity for a few years. We use numerical examples to illustrate new methods for solving this problem.*

***Keywords:*** *Aggregation, Decomposition, Labor productivity, Logarithmic mean.*

***JEL classification*** ***:*** *C61, D24, J24, O47*

[1]Independent Marketing Consultant, shu-tsuchida@nifty.com

*International Journal of Contemporary Economics and Administrative Sciences*
*ISSN: 1925 – 4423*
*Volume :5, Issue:3-4, Year:2015, pp.109 – 138*

**1. Introduction**

Labor productivity is a key indicator used in economic analysis and is sometimes computed by agencies, organizations, and scholars, including the OECD, ILO, and BLS (see, OECD 2001, 2015; ILO 2014; BLS 2008; Dean 1999; Jorgenson and Stiroh 2001; Schreyer and Pilat 2001; Pilat and Schreyer 2004; Timmer et al. 2007, 2010; Holman et al. 2008; O'Mahony and Timmer 2009; Lewrick et al. 2014; Diewert 2015; Reinsdorf 2015). In this study, we systematically derive many aggregative formulae for labor productivity growth at a firm-level, to which little attention has been given. These derivations are one of the aims of this study.

Using one of these formulae, we also examine the link between labor productivity growth (or decline) at the firm-level and that at the industry-level (i.e., total labor productivity growth), and identify the situations in which the former is not correlated with the latter. Regarding this link, we expose some important properties required for an aggregative formula. From these discussions and on examination of all the derived aggregations, we formulate a strategic decomposition that is a central concept of this study. The decomposition tells us that firm labor productivity in the base period is a misleading signal to increase total labor productivity.

Strategic decomposition can solve the problem of maximizing the increase in total labor productivity for a few years. Thus, we illustrate methods of solving this problem using strategic decomposition. The derivation and applications of the strategic decomposition are main aims of this study.

The labor productivity of the $i$th firm at period $t$ ($t$=0, 1, 2, …), $p_{it}$, is defined by $p_{it} = y_{it}/h_{it}$, where $y_{it} > 0$ and $h_{it} > 0$ are, respectively, the real output[2] and labor input (total hours worked) of a firm during a certain period. Labor productivity at the industry-level including such the firm, $P_t$, is then defined by $P_t = Y_t/H_t$, where $Y_t$ denotes the real output of the industry and $H_t$ the labor input. Herein, $Y_t = \Sigma y_{it}$ and $H_t = \Sigma h_{it}$. (In this paper, the summation ($\Sigma$) and product ($\Pi$) are always made over all firms belonging to the same industry as the subject firm, so the indexes of those are suppressed. Besides, $y_t$ is measured by netting out all the inter-firm transactions in that industry.)

Simple aggregative formulae of labor productivity at period $t$ are:

$$P_t = \frac{\Sigma y_{it}}{H_t} = \frac{\Sigma h_{it}\,(y_{it}/h_{it})}{H_t} = \sum{}' b_{it} p_{it} \tag{1}$$

$$\frac{1}{P_t} = \frac{\Sigma h_{it}}{Y_t} = \frac{\Sigma y_{it}\,(h_{it}/y_{it})}{Y_t} = \sum{}' a_{it}(1/p_{it}) \tag{2}$$

wherein $a_{it} = y_{it}/Y_t$ and $b_{it} = h_{it}/H_t$ are, respectively, the $i$th firm's shares of real output and labor input in that industry. The aggregation (1) is the weighted arithmetic mean of $p_{it}$, the sum of which weights, $b_{it}$, equals unity. In contrast, the aggregation (2) is the weighted harmonic mean of $p_{it}$, the sum of which weights, $a_{it}$, also equals unity. As shown in the two equations, two shares link productivity at the industry-level and that at the firm-level.

[2] Thus, we do not consider each individual price. For the deflators, see Reinsdorf (2015).

These shares will play a key role later. In addition, $a_{it}/b_{it} = p_{it}/P_t$ always holds for any *i*th firm at period *t* and will be used repeatedly. We elaborate upon the aggregation for the growth or decline of firms' labor productivity. That is, we elaborate upon the total labor productivity quotient (TLPQ), $P_{t+1}/P_t$, using these shares and decompose it into the contributions of individual firm.[3] We also discuss total labor productivity rate (TLPR) $(P_{t+1}-P_t)/P_t$.

A familiar decomposition of TLPR, which will be exhibited in Subsection 5.1, is shown as the formula in which the numerator ($P_{t+1}$ and $P_t$) is derived by the extreme right of (1) and its denominator is given by $P_t = Y_t/H_t$ (for example, Tang and Wang 2004; Vijselaar and Albers 2004; De Avillez 2012; Dumagan 2013). Because this formula use only $b_{it}$ at first, it suffers a serious defect that will be explained later. We do not agree with its use. As mentioned above, the two shares, $a_{it}$ and $b_{it}$, are equivalently important.

Here we refer to an aggregation whereby either $a_{it}$ or $b_{it}$ (or, $L(a_i)$ or $L(b_i)$) is employed to derive TLPQ (or logarithmic TLPQ) at the first procedure as a *single-handed aggregation*, and to one whereby both $a_{it}$ and $b_{it}$ are employed to do so as a *double-handed aggregation*. After that, the formula may include both $a_{it}$ and $b_{it}$. Later, $L(a_i)$ and $L(b_i)$ will be defined. (For $a_{it}/b_{it}$, $L(a_i/b_i)$, etc., see Appendix A.)

The remainder of this paper is organized as follows. In Section 2, we discuss an additive formula of TLPQ, which includes two formulae; single-handed and double-handed. This section shows a generalized aggregation. In Section 3, we discuss a multiplicative formula of TLPQ, which also includes two formulae. We also explain a logarithmic mean that is indispensable to derive a multiplicative formula. In Section 4, we then express the relationships between labor productivity at the firm-level and that at the industry-level. In Section 5, we explain the decomposition of TLPR. The explanation involves showing both the familiar decomposition above and an approximately generalized decomposition. In Section 6, we explain a strategic decomposition that can be used to select the most important firm to perform the largest TLPR in a certain industry. Because our strategic decomposition is approximated, we need to compare a strategy derived using this decomposition with one derived by the generalized aggregation. We discuss this important subject in this section using specific assumptions. The strategic decomposition can also be used to solve the problem of needing to maximize the increase of the TLPR for a few years. We thus illustrate some applications using numerical examples in Section 7. This is the most informative section of the paper. Finally, Section 8 presents some concluding remarks.

## 2. Total labor productivity quotient: an additive formula

### *2.1. Fundamental relations*

Our approach to derive an aggregative formula requires some fundamental relations. To derive an additive formula, we use:

[3] Our analysis can also apply to the link between the quotient of labor productivity at the overall economy level and those at the industry-level.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

$$\frac{P_1}{P_0}=\frac{Y_1/Y_0}{H_1/H_0}=\frac{H_0/H_1}{Y_0/Y_1}, \tag{3}$$

$$\frac{Y_1}{Y_0}=\sum{}^{'}\frac{y_{i1}}{Y_0}\left(\frac{y_{i0}}{y_{i0}}\right)=\sum{}^{'}a_{i0}\left(\frac{y_{i1}}{y_{i0}}\right),\frac{Y_0}{Y_1}=\sum{}^{'}a_{i1}\left(\frac{y_{i0}}{y_{i1}}\right),$$
$$\frac{H_1}{H_0}=\sum{}^{'}\frac{h_{i1}}{H_0}\left(\frac{h_{i0}}{h_{i0}}\right)=\sum{}^{'}b_{i0}\left(\frac{h_{i1}}{h_{i0}}\right),\frac{H_0}{H_1}=\sum{}^{'}b_{i1}\left(\frac{h_{i0}}{h_{i1}}\right), \tag{4}$$

where the subscript 1 represents the comparison period and 0 the base period.

*2.2. Single-handed aggregative formula*

We show two aggregations. One is derived by substituting $Y_1/Y_0$ in (4) into that in (3). (Note that we do not substitute $H_1/H_0$ or $H_0/H_1$ in (4) into that in (3).) This aggregation is

$$\frac{P_1}{P_0}=\left(\sum{}^{'}a_{i0}\left(\frac{y_{i1}h_{i0}}{y_{i0}h_{i1}}\right)\left(\frac{h_{i1}}{h_{i0}}\right)\right)\left(\frac{H_0}{H_1}\right)=\sum{}^{'}a_{i0}\left(\frac{b_{i1}}{b_{i0}}\right)\left(\frac{p_{i1}}{p_{i0}}\right)=\left(\frac{P_1}{P_0}\right)\sum{}^{'}a_{i1} \tag{5}$$

(The last equations of (5) and (6) are disregarded below, though we use them in Appendix B.) If $Y_t=\Sigma\eta_{it}y_{it}$ where $\eta_{it}$ is the $i$th firm's real output price as shown in Diewert (2015), similar procedures deriving (5) will produce (9) in Diewert (2015) (see also our (13) below). Substituting $H_0/H_1$ in (4) into that in (3), we obtain the other aggregation:

$$\frac{P_1}{P_0}=\left(\sum{}^{'}b_{i1}\left(\frac{h_{i0}y_{i1}}{h_{i1}y_{i0}}\right)\left(\frac{y_{i0}}{y_{i1}}\right)\right)\left(\frac{Y_1}{Y_0}\right)=\sum{}^{'}b_{i1}\left(\frac{a_{i0}}{a_{i1}}\right)\left(\frac{p_{i1}}{p_{i0}}\right)=\left(\frac{P_1}{P_0}\right)\sum{}^{'}b_{i0}. \tag{6}$$

Thus TLPQ, $P_1/P_0$, becomes the weighted arithmetic mean of the quotients of labor productivity at the firm-level, $p_{i1}/p_{i0}$. The sums of the weights in (5) and (6) are, respectively, $\Sigma a_{i0}(b_{i1}/b_{i0})=(\Sigma b_{i1}p_{i0})/(\Sigma b_{i0}p_{i0})$ and $\Sigma b_{i1}(a_{i0}/a_{i1})=(\Sigma a_{i0}/p_{i1})/(\Sigma a_{i1}/p_{i1})$. These ordinarily do not equal unity, because $P_1/P_0=1$ does not have to hold even if $p_{i1}/p_{i0}=1$ for all $i$. This material fact is illustrated in Table 1 in Subsection 7.2 below.

Each term, $a_{i0}(b_{i1}/b_{i0})(p_{i1}/p_{i0})$ in (5) and $b_{i1}(a_{i0}/a_{i1})(p_{i1}/p_{i0})$ in (6), is considered the contribution of the $i$th firm to the TLPQ. Hence, a firm can contribute to TLPQ without doing anything. That is, the firm that changes neither real output nor labor input can contribute to TLPQ. We call this a "*basic property*" that an aggregative formula must possess. The property may not presently be widely accepted and thus we will repeatedly examine this point later. As seen in (5) and (6), the firm's contribution to the TLPQ differs according to the aggregative formulae. Furthermore, this contribution also varies with every change of the deflator used (see, for example, De Avillez 2012).

Two aggregations exhibit a serious defect; that is, the firms' contributions to the TLPQ are independent of their input or output changes. We call this defect the invariable contribution (IN-C). All single-handed aggregations below cause the IN-C. Since the defect is not our main concern, it will be discussed further in Appendix B.

*2.3. Double-handed aggregative formula (the generalized aggregation)*

To derive a double-handed formula, we must substitute each term (e.g., $Y_1/Y_0$, $H_1/H_0$, and so on) in (4) into that in (3). These procedures lead to two forms:

$$\frac{P_1}{P_0} = \frac{(Y_1/Y_0)}{(H_1/H_0)} = \frac{\Sigma a_{i0}(y_{i1}/y_{i0})}{\Sigma b_{i0}(h_{i1}/h_{i0})} \tag{7}$$

$$\frac{P_1}{P_0} = \frac{(H_0/H_1)}{(Y_0/Y_1)} = \frac{\Sigma b_{i1}(h_{i0}/h_{i1})}{\Sigma a_{i1}(y_{i0}/y_{i1})} \tag{8}$$

These also tell us that a firm can contribute to TLPQ without doing anything. Although these aggregations cannot decompose the TLPQ into individual firm contributions, these approximations that will be shown as (21) and (23) below can. Since the aggregations have the requisite basic properties and are free from the IN-C, we regard them as *suitable aggregations*. We call (7) the generalized aggregation for a simple reference. We can derive other formulae, one of which causes the IN-C. These other formulae are shown in Appendix A.

## 3. Total labor productivity quotient: a multiplicative formula

*3.1. Log-change form and a logarithmic mean*

A multiplicative formula written in log-change variables employs two differences of any positive variables. These variables, namely $X_t$ ($t$ = 0 and 1), are given by $\Delta X = X_1 - X_0$ and $\Delta \log X = \log X_1 - \log X_0 = \log(X_1/X_0)$, where the subscript 1 and 0 are the same as above and the logarithm is natural.

Because a logarithmic mean is indispensable to the derivation of a multiplicative form, we briefly comment on it (see Carlson 1972; Stolarsky 1975; Vartia 1976; Tsuchida 1997, 2014; Balk 2008). The logarithmic mean, $L(X)$, is defined by

$$L(X) \equiv \frac{\Delta X}{\Delta \log X} = \frac{X_1 - X_0}{\log X_1 - \Delta \log X_0} = \frac{X_0 - X_1}{\log X_0 - \log X_1}$$

The value of $L(X)$ is always positive and has the limit:

$$\lim_{\Delta X \to 0} L(X) = X_1 = X_0.$$

If $X_1/X_0$ is also close to 1, it can be approximated by the usual three means: arithmetic, geometric, and harmonic (see Tsuchida 1997, 2014).

Letting $X_t = \Sigma x_{it}$, $x_{it} > 0$, and $w_{it} = x_{it}/X_t$, an inequality $L(X) \geqq \Sigma L(x_i)$ holds. Moreover, there exists an aggregation property $\Sigma L(w_i)\Delta \log w_i = \Sigma \Delta w_i = 0$, which is valuable in this paper.

*3.2. Fundamental relations*

We show some fundamental relations that will be repeatedly used later. The log-change labor productivity quotient at the firm-level is

$$\Delta \log p_i = \Delta \log y_i - \Delta \log h_i \tag{9}$$

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

and that at the industry-level is

$$\Delta\log P = \Delta\log Y - \Delta\log H. \qquad (10)$$

Because $a_{it} = y_{it}/Y_t$ and $b_{it} = h_{it}/H_t$, we have $\Delta\log a_i = \Delta\log y_i - \Delta\log Y$ and $\Delta\log b_i = \Delta\log h_i - \Delta\log H$. Thus, we produce the log-change aggregations as follows:[4]

$$\Delta\log Y = \sum\nolimits' \left(L(a_i)/\left(\sum\nolimits' L(a_j)\right)\right)\Delta\log y_i = \sum\nolimits' \lambda(a_i)\Delta\log y_i \qquad (11)$$

and

$$\Delta\log H = \sum\nolimits' \left(L(b_i)/\left(\sum\nolimits' L(b_j)\right)\right)\Delta\log h_i = \sum\nolimits' \lambda(b_i)\Delta\log h_i \qquad (12)$$

where $\lambda(a_i) = L(a_i)/(\Sigma L(a_j))$, $\Sigma\lambda(a_i) = 1$, $\lambda(b_i) = L(b_i)/(\Sigma L(b_j))$, and $\Sigma\lambda(b_i) = 1$. Equation (11) can be found in Balk (2014, eq. (7)).[5]

*3.3. Single-handed aggregative formula*

Using (9), (10), and (11), we have

$$\Delta\log P = \Delta\log Y - \log H$$
$$= \sum\nolimits' \lambda(a_i)\left(\Delta\log y_i - \Delta\log h_i + \Delta\log h_i - \Delta\log H\right),$$

from which we get

$$\Delta\log P = \sum\nolimits' \lambda(a_i)\Delta\log p_i + \sum\nolimits' \lambda(a_i)\Delta\log b_i. \qquad (13)$$

Note that we do not use (12). This corresponds to the additive aggregation (5) (compare the derivation procedure of (5) with that of (13)). If the logarithmic mean can be approximated by the arithmetic mean $A(a_i)$, that is, $L(a_i) \approx A(a_i) \equiv (a_{i1} + a_{i0})/2$, then we have $\lambda(a_i) = A(a_i)$. Hence, the approximation of (13) is

$$\Delta\log P \approx \sum\nolimits' A(a_i)\Delta\log p_i + \sum\nolimits' A(a_i)\Delta\log b_i. \qquad (14)$$

This formula is the same as (7) in Stiroh (2002) and (5.4) in Timmer et al. (2010).

Similar procedures using (9), (10), and (12) produce the following multiplicative aggregation that corresponds to the additive aggregation (6):

$$\Delta\log P = \sum\nolimits' \lambda(b_i)\Delta\log p_i - \sum\nolimits' \lambda(b_i)\Delta\log a_i.$$

Since all these formulae are the single-handed aggregations, they give rise to the IN-C above, some of which are explained in Appendix B.

[4] Notably, $\Sigma L(a_i)\Delta\log a_i = \Sigma\Delta a_i = 0$ and $\Sigma L(b_i)\Delta\log b_i = \Sigma\Delta b_i = 0$.
[5] We disregard the difference between real and nominal output.

*3.4. Double-handed aggregative formula (the well-known log-change aggregation)*

Using (10), (11), and (12), we can quickly derive a double-handed aggregation as follows:

$$\Delta \log P = \sum{}' \lambda(a_i) \Delta \log y_i - \sum{}' \lambda(b_i) \Delta \log h_i. \tag{15}$$

This formula can be found in Balk (2014, eq. (19)). If each logarithmic mean can be approximated by its arithmetic mean, then we have

$$\Delta \log P \approx \sum{}' A(a_i) \Delta \log y_i - \sum{}' A(b_i) \Delta \log h_i. \tag{16}$$

While the formulae (15) and (16) have properties similar to the aggregation (A.43) in OECD (2001), our (15) is an exact and appropriate formula for discrete data. Below, we call (16) the well-known log-change aggregation (see also OECD 2015).

Equation (15) is rewritten as

$$\Delta \log P = \sum{}' \lambda(a_i) \Delta \log p_i + \sum{}' \big(\lambda(a_i) - \lambda(b_i)\big) \Delta \log h_i, \tag{17}$$

which corresponds to the equation (A.44) in OECD (2001). Given some assumptions, we have approximations such as $\lambda(a_i) \approx a_{i0}$ and $\lambda(b_i) \approx b_{i0}$ (see footnote 12 below). Thus, our (17) approaches the equation (1) in Nordhaus (2002). These three equations, (15), (16), and (17), are not suitable aggregations, because the $i$th firm without doing anything cannot contribute to the logarithmic TLPQ, in which $\Delta \log y_i = 0$ and $\Delta \log h_i = 0$ hold.[6] That is, these three equations lack the basic property. Provided that we use other logarithmic means, we can derive suitable aggregations that do not suffer this fault. Refer to (34), (35), and so on in Appendix A.

## 4. Labor productivity at the firm-level and that at the industry-level

Hereafter, we write any $i$th firm's output and input in the comparison period as $y_{i1} = y_{i0}(1 + u_i)$ and $h_{i1} = h_{i0}(1 + v_i)$, given the assumptions $-1 < u_i < 1$ and $-1 < v_i < 1$. These $u_i$ and $v_i$ produce a simple form as shown below. We also use the reverse assumptions, such as $y_{i0} = y_{i1}(1 + u_i^*)$ and $h_{i0} = h_{i1}(1 + v_i^*)$. These $u_i$ and $v_i$ (or $u_i^*$ and $v_i^*$) are considered control variables in the problem of how to increase TLPR, which will be discussed later. (In Appendix B, $u_i$ and $v_i$ play main roles.)

Consider the relationships between labor productivity at the firm-level and that at the industry-level using the generalized aggregation (7). For a simple discussion, there are two firms (the $j$th and $k$th firms) in a certain industry and the $j$th firm alone changes its output and input. Thus, $u_j \neq 0, v_j \neq 0,$ and $u_k = v_k = 0$.

*4.1. Unconventional findings*

Each firm's labor productivity quotient, $q_i$ ($i = j$ and $k$), is

[6] Note that $y_{i0} = y_{i1}$, $h_{i0} = h_{i1}$, and $p_{i0} = p_{i1}$.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

$$q_j = \frac{p_{j1}}{p_{j0}} = \frac{1+u_j}{1+v_j}, q_k = \frac{p_{k1}}{p_{k0}} = 1.$$

The TLPQ, $Q$, is

$$Q = \frac{P_1}{P_0} = \frac{a_{j0}(1+u_j) + a_{k0}}{b_{j0}(1+v_j) + b_{k0}} = \frac{1 + a_{j0}u_j}{1 + b_{j0}v_j} \quad (18)$$

because $\Sigma a_{i0} = 1$ and $\Sigma b_{i0} = 1$. Although (18) is the aggregative formula for this simple case, it cannot compute each firm's contribution to the TLPQ.

The aggregation (18) guides us to unconventional findings. The $j$th firm's labor productivity growth $q_j > 1$ (or decline $q_j < 1$) is not necessarily consistent with the total labor productivity growth $Q > 1$ (or decline $Q < 1$). (Also see a "productivity paradox" explained in Fox (2012).) The ratio $a_{j0}/b_{j0} = p_{j0}/P_0$ plays an important role. If, generally speaking, the firm's productivity $p_{j0}$ is very low compared to the total productivity $P_0$, the growth (or decline) of the former may lead to the decline (or growth) of the latter. Moreover, even if $u_j = v_j$, that is $p_{j1} = p_{j0}$, the total productivity may grow or decline. Refer to a similar concept called the Denison Effect by Nordhaus (2002), and see also the next subsection and Section 7. The crucial factors ($a_{j0}$ and $b_{j0}$) depend upon the $k$th firm's output and input at the base period. Thus, the $k$th firm can indirectly contribute to the TLPQ without doing anything[7], though this basic property is repeatedly discussed above.

*4.2. Short run production function and a u-v ratio*

The Cobb-Douglas production function for the $i$th firm is:

$$\log y_{it} = \log c_{it} + \alpha_i \log k_{it} + \beta_i \log h_{it},$$

where $c_{it}$ is total factor productivity (TFP) and $k_{it}$ capital input (adjusted by its degree of utilization); and $\alpha_i$ and $\beta_i$ are the output elasticities of capital and labor, respectively. In the short run, we may assume that $c_{it}$ and $k_{it}$ are constants or their log-change values are proportional to that of labor input; consequently

$$\Delta \log c_i + \alpha_i \Delta \log k_i = \gamma_i \Delta \log h_i$$

wherein $\gamma_i$ is a constant or null. Thus the short run production function is

$$\Delta \log y_i = \varepsilon_i \Delta \log h_i$$

wherein $\varepsilon_i = \beta_i + \gamma_i$. If a positive variable $x$ approaches 1, we can use an approximation given by $\log x \approx x - 1$. We call this the log-approximation and will often use it later. Applying the approximation to the above, we have

$$u_i = \varepsilon_i v_i \quad (19)$$

[7] Strictly speaking, this is true only if $u_j \neq 0$ or $v_j \neq 0$. Putting it another way, facing a stagnating business, the $k$th firm made an efficient choice, but the $j$th firm did not. The choices of these firms resulted in $u_k = v_k = 0$, $u_j < 0$, and $v_j < 0$; and the total labor productivity thus increased.

and we call $\varepsilon_i$ the *i*th firm's u-v ratio. While (19) is an approximation, for simplicity we consider it the equation in the present discussion.[8]

Given that the foregoing *j*th firm has the just mentioned production function, we have:

$$\frac{\partial q_j}{\partial v_j} = \frac{\varepsilon_j - 1}{(1+v_j)^2}.$$

$$\frac{\partial Q}{\partial v_j} = \frac{\varepsilon_j a_{j0} - b_{j0}}{(1+b_{j0}v_j)^2}.$$

These equations provide the following additional findings:

$$\text{If } \varepsilon_j \gtreqless 1, \text{ then } \frac{\partial q_j}{\partial v_j} \gtreqless 0,$$

$$\text{If } \varepsilon_j a_{j0} \gtreqless b_{j0}, \text{ then } \frac{\partial Q}{\partial v_j} \gtreqless 0.$$

Thus, $a_{j0}$, $b_{j0}$, and $\varepsilon_j$ clarify relationships between the labor productivity quotient at the firm-level and that at the industry-level. We should recognize that in many situations productivity growth (or decline) at the firm-level does not correlate with that at the industry-level (see Table 1 and Table 2 below).

## 5. Decomposition of total labor productivity rate

In this section, we explain the decomposing form of only TLPR ($\Delta P/P_0$), since our main aims are the strategic decomposition and its application, which will be discussed later.

### *5.1. Single-handed formula (the familiar decomposition)*

We can derive $\Delta P$ from (1) as follows:

$$\Delta P = P_1 - P_0 = \Sigma b_{i1}p_{i1} - \Sigma b_{i0}p_{i0}.$$

This relation and $P_0$ lead to the equation regarded as the familiar decomposition:

$$\frac{\Delta P}{P_0} = \frac{\Sigma b_{i1}p_{i1} - \Sigma b_{i0}p_{i0}}{P_0} = \frac{\Sigma b_{i0}\Delta p_i + \Sigma p_{i1}\Delta b_i}{P_0} = \sum\nolimits' \left(\frac{a_{i0}\Delta p_i}{p_{i0}} + \frac{a_{i0}p_{i1}\Delta b_i}{b_{i0}p_{i0}}\right), \quad (20)$$

wherein we employed the relation $P_0 = (b_{i0}p_{i0})/a_{i0}$ for the denominator (remember that each individual price is not considered). The last equation of (20) is the same as the equations (3) or (5) in Tang and Wang (2004)[9] (see also Vijselaar and Albers 2004; De Avillez 2012; Dumagan 2013; Reinsdorf 2015). The form designated as "shift-share

[8] We can directly use (19) as the assumption without the other assumptions above.

[9] Our derivation of (20) is easier than theirs. Note that $p_{i1} = p_{i0} + \Delta p_i$.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

analysis" of TLPR in European Commission (2004, p. 155) becomes the same as (20).[10] Since (20) is the single-handed aggregation, this yields the IN-C that is explained in Appendix B.

*5.2. Double-handed formula*

Applying the log-approximation to $\log(P_1/P_0)$, $\log(Y_1/Y_0)$, and $\log(H_1/H_0)$ in (10), we have $(P_1/P_0) - 1 = \Delta P/P_0 \approx (Y_1/Y_0) - (H_1/H_0)$. From these and (7), we obtain a formula that is regarded as an approximately generalized decomposition:

$$\frac{\Delta P}{P_0} \approx \sum{}' \left(a_{i0}(y_{i1}/y_{i0}) - b_{i0}(h_{i1}/h_{i0})\right) = \sum{}' \left(a_{i0}(1+u_i) - b_{i0}(1+v_i)\right). \tag{21}$$

The contribution of the *i*th firm to the TLPR is about $a_{i0}(1+u_i) - b_{i0}(1+v_i)$ and is dependent upon the two control variables.

Because $\Sigma a_{i0} = 1$ and $\Sigma b_{i0} = 1$, we have

$$\frac{\Delta P}{P_0} \approx \sum{}' (a_{i0}u_i - b_{i0}v_i). \tag{22}$$

Note that (22) is not the form used to calculate each firm's contribution to the TLPR. The question thus arises of the meaning of the term $a_{i0}u_i - b_{i0}v_i$. For a minute, we call this the component of the *i*th firm.

Because (22) leads to our central concept below, we show another derivation using more permissible assumptions. Then, (7) produces

$$\frac{\Delta P}{P_0} = \frac{(1+\Sigma a_{i0}u_i)(1-\Sigma b_{i0}v_i)}{(1+\Sigma b_{i0}v_i)(1-\Sigma b_{i0}v_i)} - 1 = \frac{\Sigma(a_{i0}u_i - b_{i0}v_i) - (\Sigma a_{i0}u_i)(\Sigma b_{i0}v_i) + (\Sigma b_{i0}v_i)^2}{1-(\Sigma b_{i0}v_i)^2}.$$

Hence, we can derive (22) provided the following assumptions hold:

$(\Sigma b_{i0}v_i)^2 = (H_1/H_0 - 1)^2 \ll 1$ and

$\mathrm{ABS}\left((\Sigma a_{i0}u_i)(\Sigma b_{i0}v_i)\right) = \mathrm{ABS}\left((Y_1/Y_0 - 1)(H_1/H_0 - 1)\right) \ll 1$.

Applying the log-approximation to (8) produces two approximations:

[10] The former is

$$\frac{\Delta P}{P_0} = \sum{}' \left(\frac{a_{i0}\Delta p_i}{p_{i0}} + \frac{p_{i0}}{P_0}(b_{i1} - b_{i0}) + \frac{\Delta p_i \Delta b_i}{P_0}\right),$$

which is also used in OECD (2014).

$$\frac{\Delta P}{P_0} \approx \sum{}' \left( b_{i1}(h_{i0}/h_{i1}) - a_{i1}(y_{i0}/y_{i1}) \right) = \sum{}' \left( b_{i1}(1+v_i^*) - a_{i1}(1+u_i^*) \right), \tag{23}$$

$$\frac{\Delta P}{P_0} \approx \sum{}' (b_{i1}v_i^* - a_{i1}u_i^*) \tag{24}$$

Some double-handed multiplicative formulae have simple approximations. Application of the log-approximation to (15) can yield (22) and (24).[11] Similarly, we have two approximations from (16):

$$\frac{\Delta P}{P_0} \approx \sum{}' \left( A(a_i)(1+u_i) - A(b_i)(1+v_i) \right), \tag{25}$$

$$\frac{\Delta P}{P_0} \approx \sum{}' (A(a_i)u_i - A(b_i)v_i) \tag{26}$$

The terms $b_{i1}v_i^* - a_{i1}u_i^*$ in (24) and $A(a_i)u_i - A(b_i)v_i$ in (26) are the component of the *i*th firm to their TLPR.

Whenever we regulate TLPR, we must distinguish two types of information, independent and not independent of control variable. We can consider the former *ex ante* information and the latter *ex post* information. Equation (22) needs *ex ante* information (two shares at the base period), (24) needs *ex post* information (two shares at the comparison period), and (26) needs both *ex ante* and *ex post* information. In the below, *ex ante* information will turn to active players.

## 6. Strategic decomposition and comparison test

### *6.1. Strategic decomposition*

Hereafter we assume $\varepsilon_i$ to be *ex ante* information. Thus we know that the *i*th firm's output will increase (or decrease) by $100\varepsilon_i v_i$ percent given an increase (or decrease) in its input by $100v_i$ percent during those years. Our decomposition (22) then becomes

$$\frac{\Delta P}{P_0} \approx \sum{}' (a_{i0}\varepsilon_i v_i - b_{i0}v_i) = \sum{}' (\varepsilon_i a_{i0} - b_{i0})v_i \,. \tag{27}$$

Now we try to clarify a *strategic decomposition* that is our central concept. We use this terminology to argue for a decomposition of TLPR that satisfies these two conditions:

S1. It must be derived from the double-handed suitable aggregation or its approximation.

S2. The component of the *i*th firm must be a function only of its *ex ante* information ($a_{i0}$, $b_{i0}$, $\varepsilon_i$, $y_{i0}$, $h_{i0}$, $p_{i0}$, and $P_0$)[12] and its control variable ($v_i$).

Although we have shown many aggregations up to here, the derivations of which are one of the aims of this paper, few aggregations satisfy the condition S1. Other aggregations

---

[11] Note that we have $L(X) \approx X_0$ or $L(X) \approx X_1$ since $L(X) = \frac{X_0(X_1/X_0-1)}{\log(X_1/X_0)} = \frac{X_1(X_0/X_1-1)}{\log(X_0/X_1)}$.

[12] Note that $P_0$ is the *i*th firm's *ex ante* information because $a_{i0}/b_{i0} = p_{i0}/P_0$.

that satisfy the condition S1 are found in Appendix A. On these grounds we have concluded that only the decomposition (27) can satisfy these two conditions. Thus we call (27) the strategic decomposition. The significance of (27) to the regulation of TLPR will be understood soon. To avoid confusion, we call $(\varepsilon_i a_{i0} - b_{i0})v_i$ in (27) the strategic component of the *i*th firm to the TLPR.

*6.2. Comparison test*

We consider a problem that requires us to select the most important firm to perform the largest TLPR in a certain industry. Hereafter we may call this solution a strategy. Because our strategic decomposition (27) is an approximated form, we need to compare the strategy derived by (27) with that by the generalized aggregation (7). However, we cannot decompose (7) in a general situation. Hence, we shall discuss the momentous subject using a specific assumption wherein a single firm (any *i*th firm) can independently change its output and input as in Section 4.

From (18), we obtain the criterion *C*:

$$\frac{\Delta P}{P_0} = \frac{P_1}{P_0} - 1 = \frac{1 + a_{i0}u_i}{1 + b_{i0}v_i} - 1 = \frac{(\varepsilon_i a_{i0} - b_{i0})v_i}{1 + b_{i0}v_i} \equiv C(a_{i0}, b_{i0}, \varepsilon_i, v_i). \tag{28}$$

One of the strategies is derived by maximizing *C* under some conditions. The strategic decomposition (27) leads to *Cs* given by

$$\frac{\Delta P}{P_0} \approx (\varepsilon_i a_{i0} - b_{i0})v_i \equiv Cs(a_{i0}, b_{i0}, \varepsilon_i, v_i) \tag{29}$$

The other strategy is derived by maximizing *Cs* under the conditions above

Employing the following six tests, each of which corresponds to $\varepsilon_i \to \infty$, $\varepsilon_i = 0$, $\varepsilon_i > 0$, or $\varepsilon_i < 0$, we examine whether or not the strategy derived by *Cs* is the same as that derived by *C*. Our discussions only involve the positive results; that is, $C > 0$ and $Cs > 0$ (the negative ones will alike be discussed).[13] Thus, there may be no strategies (or solutions).

[1] Comparison test 1: $\varepsilon_i \to \infty$, $u_i = x$, $0 < x < 1$, and $v_i = 0$ for any *i*th firm.[14]

These conditions imply that any firm can increase its output by100 *x* percent without changing its input. (Similar things are implied below.) As in the controlled economy, we consider a strategy wherein a firm is selected to execute the largest (or near largest) TLPR.

In this test, the values of our strategic decomposition (29) and the criterion (28) are given by

$$Cs = a_{i0}x \text{ and } C = a_{i0}x.$$

Maximizing *Cs* is one of our strategies and tells us that we must select a firm whose $a_{i0}$ is the highest in that industry. We write this as follows:

[13] Note that *C* and *Cs* have the same sign.
[14] The conditions are obtained because $v_i = u_i / \varepsilon_i$.

$$\text{Max of } Cs \Rightarrow \text{Max } \{a_{i0}\}.$$

(Given a group of firms, we may select any one of them.)

Similarly, the other strategy derived by $C$ is

$$\text{Max of } C \Rightarrow \text{Max } \{a_{i0}\}$$

The two strategies are the same and both yield a value that equals $a_{i0}x$.

[2] Comparison test 2: $\varepsilon_i = 0$, $v_i = -x$, and $0 < x < 1$ for any $i$th firm.

$$Cs = b_{i0}x \text{ and } C = b_{i0}x/(1 - b_{i0}x).$$

Thus, we have

$$\text{Max of } Cs \Rightarrow \text{Max } \{b_{i0}\}.$$

The criterion yields the same strategy, because $C$ is a strictly increasing function of $b_{i0}$.

However, the value computed by (29) is an approximation of that computed by (28). That is; if the $m$th firm is selected, the ratio of these values is

$$\frac{b_{m0}x}{b_{m0}x/(1 - b_{m0}x)} = 1 - b_{m0}x \approx 1,$$

where in practice $x$ is a small value (e.g., $0 < x \leq 0.05$).

[3] Comparison test 3: $\varepsilon_i = 1$, $v_i = x$, and $0 < x < 1$ for any $i$th firm.

$$Cs = (a_{i0} - b_{i0})x \text{ and } C = (a_{i0} - b_{i0})x/(1 + b_{i0}x),$$

which derives

$$\text{Max of } Cs \Rightarrow \text{Max } \{(a_{i0} - b_{i0}) > 0\},$$

$$\text{Max of } C \Rightarrow \text{Max } \{(a_{i0} - b_{i0})x/(1 + b_{i0}x) > 0\}.$$

While we can quickly gain the strategy for Max of $Cs$, we cannot do likewise for Max of $C$. (Note that the latter strategy is dependent on $x$ which is unknown. The same holds in some of the tests below.)

In this test, the strategy derived by $Cs$ may not be the same as that derived by $C$. Suppose that the $m$th firm is selected by $Cs$ and the $k$th firm by $C$. We then have the inequality:

$$1 \leq \frac{(a_{m0} - b_{m0})x}{(a_{k0} - b_{k0})x} \leq \frac{1 + b_{m0}x}{1 + b_{k0}x} \approx 1 + (b_{m0} - b_{k0})x \approx 1,$$

wherein $b_{m0} \geq b_{k0}$. Hence the value of the $m$th firm calculated by (28) nearly equals that of the $k$th firm calculated by $C$. (Remember that we cannot practically solve the Max of $C$.)

If $a_{i0} = b_{i0}$ for all $i$, there are no solutions since $C = 0$ and $C_s = 0$. (We may have a similar case in test 4 below.) We strongly emphasize that Max $\{(a_{i0} - b_{i0}) > 0\}$ differs from Max

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

$\{(a_{i0}/b_{i0}) = (p_{i0}/P_0) > 1\}$. So, a firm with the highest productivity in its industry at the base period may not be selected.[15]

[4] Comparison test 4: $\varepsilon_i = 1$, $v_i = -x$, and $0 < x < 1$ for any $i$th firm.

$$Cs = (b_{i0} - a_{i0})x \text{ and } C = (b_{i0} - a_{i0})x/(1 - b_{i0}x),$$

from which we derive

$$\text{Max of } Cs \Rightarrow \text{Max } \{(b_{i0} - a_{i0}) > 0\}.$$

While two strategies are not identical, the difference between two values calculated by (28) is small since the analogous inequalities hold as in test 3. Additionally, Max $\{(b_{i0} - a_{i0}) > 0\}$ differs from Max $\{(b_{i0}/a_{i0}) = (P_0/p_{i0}) > 1\}$.

[5] Comparison test 5: $\varepsilon_i = -1$, $v_i = -x$, and $0 < x < 1$ for any $i$th firm.

$$Cs = (a_{i0} + b_{i0})x \text{ and } C = (a_{i0} + b_{i0})x/(1 - b_{i0}x),$$

which derives

$$\text{Max of } Cs \Rightarrow \text{Max } \{(a_{i0} + b_{i0})\}.$$

Here, $Cs$ also leads to the near-maximum value, since the analogous inequalities as in test 3 hold.

[6] Comparison test 6: $\varepsilon_i = -1$, $v_i = x$, and $0 < x < 1$ for any $i$th firm.

$$Cs = -(a_{i0} + b_{i0})x < 0 \text{ and } C = -(a_{i0} + b_{i0})x/(1 + b_{i0}x) < 0$$

No solutions exist for the above equations because $Cs < 0$ and $C < 0$ always hold.

As explained in these tests, our strategy derived by (29) can easily be obtained and yields the value that equals or nearly equals the maximum value computed by (28). In contrast, the strategy derived by criterion (28) may not be obtained. Furthermore, consider the problem of having to select a few firms to increase the TLPR by $z$ present. Consequently we cannot use criterion (28). Our strategic decomposition is also helpful for solving such complex problems.

## 7. Strategic decomposition applied to numerical examples

Assuming firms under the planned economy, we consider circumstances in which a government must regulate TLPR at a certain industry. Our problem then is how to increase the TLPR as much as possible within a few years (e.g., 2 years), a problem that we can solve using strategic decomposition. Therefore, we illustrate methods of solving the problem using simple numerical examples.

Five firms ($j$, $k$, $l$, $m$, and $n$) are involved in the industry and the initial values (that is, *ex ante* information), $y_{i0}$, $h_{i0}$, $a_{i0}$, $b_{i0}$, and so on, which have proper units, are listed in the tables below. Since $\varepsilon_i$, which is assumed to be a positive constant for simple discussions, is *ex*

[15] Compare the $i$th firm having $a_{i0} = 0.6$, $b_{i0} = 0.3$ and $(a_{i0}/b_{i0}) = 2$ with the $j$th firm having $a_{j0} = 0.06$, $b_{j0} = 0.02$ and $(a_{j0}/b_{j0}) = 3$.

*ante* information, our control variable is the five-vector of $\{v_i\}$. We explain two methods to solve the problem: single-stage and multi-stage. Because the strategic decomposition (27) is the central concept in this paper, the focus of discussion is on how to use the single-stage method wherein (27) is ruling to our procedures. Employing the single-stage method, we can solve the problem immediately, which is very useful for practical purposes. However, in several cases this may lead to solutions that are nearly correct rather than correct. To obtain correct solutions, we may employ the multi-stage method. Because it is beyond the scope of this study, we shall explain only a few examples wherein the multi-stage method offers advantages over the single-stage one.

*7.1. Essential strategy*

To increase the TLPR, our strategic decomposition (27) tells us the essential strategies:

- if $\varepsilon_i a_{i0} - b_{i0} > 0$, then $v_i > 0$,
- if $\varepsilon_i a_{i0} - b_{i0} < 0$, then $v_i < 0$,
- if $\varepsilon_i a_{i0} - b_{i0} = 0$, then $v_i = 0$.

Thus, all strategic components become non-negative. Provided the range or value of the control vector $\{v_i\}$ for a certain period is either given or somehow determined, we can quickly solve the problem. This assumption is applied in the numerical examples. These strategies are practical for the government or its agency to regulate total productivity at any industry. (These strategies may be able to apply to the overall economy.)

When $\varepsilon_i = 1$ for all $i$, the strategies demand that we make the $i$th firm that meets $a_{i0}/b_{i0} = p_{i0}/P_0 > 1$ increase its input and make the $j$th firm that meets $a_{j0}/b_{j0} = p_{j0}/P_0 < 1$ decrease its input. This results in reallocation of labor (or workforce) from the lower-productivity firm to the higher-productivity firm. Notably, lower-productivity (or higher-productivity) does not indicate $p_{j0} < 1$ (or $p_{i0} > 1$). Compare this reallocation with that called "dynamic allocative efficiency" by Haltiwanger (2011). Additionally, productivity differences across firms may produce total productivity growth. Regarding these topics, see the survey by Syverson (2011) (see also Berthou and Standoz, 2014).

*7.2. Single-stage method*

We adopt the essential strategies above. Case 1 shows a nearly correct solution (the correct one is derived by the multi-stage method below). Only the single-stage method can quickly solve Cases 2 and 3.

[1] Case 1: $\varepsilon_i = 1, v_i = x, \text{and} -0.05 \le x \le 0.05$ for all the $i$th firms.

By making all the $i$th ($i = j, k, \ldots, n$) firms that satisfy $a_{i0} - b_{i0} > 0$ (or $a_{i0} - b_{i0} < 0$) increase (or decrease) their inputs by 5%, we can obtain the results shown in Table 1. The column exhibited by "$v$" tells the strategies. The TLPR increases by about 0.99%; nevertheless, all $p_{i1}$'s remain unchanged.[16] Hence, productivity growth at the industry-level does not correlate with that at any firm-level.

The columns exhibited by "StrCom" and "GenDec" indicate, respectively, each firm's strategic component and contribution to the TLPR. The former are calculated by (27) and

[16] Since $\varepsilon_i = 1$, $q_i = 1$ for all $i$. See also Table 4 and Table 5 below.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

the latter by (21). Neither the total equals the TLPR above, since they are approximations. (The tables below exhibit the same thing.) From the values in StrCom, we can see which firm is important to increasing the TLPR. The most important firm will be the *m*th firm, followed by the *j*th. While the *l*th firm's productivity at the base period is 1.20, its input is not increased.

Table 1 Regulated results of Case 1

| Firm | $y_0$ ($Y_0$) | $h_0$ ($H_0$) | $p_0$ ($P_0$) | $y_1$ ($Y_1$) | $h_1$ ($H_1$) | $p_1$ ($P_1$) | $q$ ($Q$) |
|---|---|---|---|---|---|---|---|
| *j* | 42 | 28 | 1.5000 | 44.10 | 29.40 | 1.50000 | 1 |
| *k* | 36 | 27 | 1.3333 | 37.80 | 28.35 | 1.33333 | 1 |
| *l* | 18 | 15 | 1.2000 | 18.00 | 15.00 | 1.20000 | 1 |
| *m* | 12 | 18 | 0.6667 | 11.40 | 17.10 | 0.66667 | 1 |
| *n* | 12 | 12 | 1.0000 | 11.40 | 11.40 | 1.00000 | 1 |
| Total | 120 | 100 | 1.2000 | 122.70 | 101.25 | 1.21185 | 1.00988 |

| Firm | $a_0$ | $b_0$ | $a_0 - b_0$ | $u$ | $v$ | StrCom | GenDec |
|---|---|---|---|---|---|---|---|
| *j* | 0.350 | 0.280 | 0.070 | 0.050 | 0.050 | 0.0035 | 0.0735 |
| *k* | 0.300 | 0.270 | 0.030 | 0.050 | 0.050 | 0.0015 | 0.0315 |
| *l* | 0.150 | 0.150 | 0.000 | 0 | 0 | 0 | 0 |
| *m* | 0.100 | 0.180 | -0.080 | -0.050 | -0.050 | 0.0040 | -0.0760 |
| *n* | 0.100 | 0.120 | -0.020 | -0.050 | -0.050 | 0.0010 | -0.0190 |
| Total | 1.000 | 1.000 | | | | 0.0100 | 0.0100 |

Note: The subscripts (*j*, *k*, *l*, *m*, *n*) are suppressed; and $q = p_1 / p_0$ and $Q = P_1 / P_0$.

What do the values exhibited in the GenDec column mean? The values in the GenDec column differ significantly from those in the StrCom column, but exploring these differences is beyond the scope of this paper.

[2] Case 2: $\varepsilon_i = 1.2$ and $v_i = \pm x$ for all the *i*th firms, and to increase the TLPR by about 2%.

Here, $x$ is an unknown variable. It is a trivial case in which any firm can independently change its input (the strategy can easily be obtained by (28)). Hence, we consider a case in which all the firms are regulated except those that meet $\varepsilon_i a_{i0} - b_{i0} = 0$. From (27), we have:

$$\sum{}' |1.2a_{i0} - b_{i0}||x| = 0.02,$$

from which we will get $x = \pm 0.0625$. The results including the strategies are exhibited in Table 2. The TLPR increases by about 1.94%. The most important firm to increasing the

TLPR is the *j*th, followed by the *k*th. Compare the values of the *m*th firm (esp. $q_m$) with those of other firms.

Table 2 Regulated results of Case 2

| Firm | $y_0$ ($Y_0$) | $h_0$ ($H_0$) | $p_0$ ($P_0$) | $y_1$ ($Y_1$) | $h_1$ ($H_1$) | $p_1$ ($P_1$) | $q$ ($Q$) |
|---|---|---|---|---|---|---|---|
| *J* | 42 | 28 | 1.5000 | 45.15 | 29.75 | 1.51765 | 1.01176 |
| *K* | 36 | 27 | 1.3333 | 38.70 | 28.69 | 1.34902 | 1.01176 |
| *L* | 18 | 15 | 1.2000 | 19.35 | 15.94 | 1.21412 | 1.01176 |
| *M* | 12 | 18 | 0.6667 | 11.10 | 16.88 | 0.65778 | 0.98667 |
| *N* | 12 | 12 | 1.0000 | 12.00 | 12.00 | 1.00000 | 1.00000 |
| Total | 120 | 100 | 1.2000 | 126.30 | 103.25 | 1.22324 | 1.01937 |
| Firm | $a_0$ | $b_0$ | $1.2a_0 – b_0$ | $u$ | $v$ | StrCom | GenDec |
| *J* | 0.350 | 0.280 | 0.140 | 0.0750 | 0.0625 | 0.0088 | 0.0787 |
| *K* | 0.300 | 0.270 | 0.090 | 0.0750 | 0.0625 | 0.0056 | 0.0356 |
| *L* | 0.150 | 0.150 | 0.030 | 0.0750 | 0.0625 | 0.0019 | 0.0019 |
| *M* | 0.100 | 0.180 | -0.060 | -0.0750 | -0.0625 | 0.0038 | -0.0763 |
| *N* | 0.100 | 0.120 | 0.000 | 0 | 0 | 0 | -0.0200 |
| Total | 1.000 | 1.000 | | | | 0.0200 | 0.0200 |

Note: The subscripts (*j*, *k*, *l*, *m*, *n*) are suppressed; and $q = p_1 / p_0$ and $Q = P_1 / P_0$.

[3] Case 3: $\varepsilon_i = 1.1$ and $v_i = f(b_{i0})$ for all the $i$th firms, and $\Sigma v_i h_{i0} = 0.050H_0$.

As in the planned economy, we consider the problem of needing to allocate graduates among individual firms in an industry. Here we assume that $v_j = 0$ if $\varepsilon_j a_{j0} - b_{j0} \leq 0$. Additionally, for simplicity it is assumed that $f(b_{i0}) = db_{i0}$, wherein $d$ is an unknown variable. We then get $d \approx 0.2877$.

The strategies are shown in Table 3. The TLPR increases by about 1.31%. The *j*th firm is the most important for increasing the TLPR, followed by the *k*th firm.

*7.3. Multi-stage method*

This method consists of five stages, each of which sees us select a single firm and determine its input. Remember that the number of firms is 5. We use *Cs* in Subsection 6.2, wherein $\varepsilon_i$ is *ex ante* information and the subscript "0" is converted into "*e*" below. (Since we must select a single firm, the strategic component calculated by (27) is no good anymore.) Given two or more firms with the largest *Cs*, we may select any one of them. When both the single-stage and multi-stage methods can be applied to solve the problem, the TLPR derived by the latter, at least, equals that derived by the former.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

Table 3 Regulated results of Case 3

| Firm | $y_0$ ($Y_0$) | $h_0$ ($H_0$) | $p_0$ ($P_0$) | $y_1$ ($Y_1$) | $h_1$ ($H_1$) | $p_1$ ($P_1$) | $q$ ($Q$) |
|---|---|---|---|---|---|---|---|
| *j* | 42 | 28 | 1.5000 | 45.72 | 30.26 | 1.51118 | 1.00745 |
| *k* | 36 | 27 | 1.3333 | 39.08 | 29.10 | 1.34294 | 1.00721 |
| *l* | 18 | 15 | 1.2000 | 18.85 | 15.65 | 1.20496 | 1.00414 |
| *m* | 12 | 18 | 0.6667 | 12.00 | 18.00 | 0.66667 | 1.00000 |
| *n* | 12 | 12 | 1.0000 | 12.00 | 12.00 | 1.00000 | 1.00000 |
| Total | 120 | 100 | 1.2000 | 127.65 | 105.00 | 1.21573 | 1.01311 |
| Firm | $a_0$ | $b_0$ | $1.1a_0 – b_0$ | $u$ | $v$ | StrCom | GenDec |
| *j* | 0.350 | 0.280 | 0.105 | 0.0886 | 0.0806 | 0.0085 | 0.0785 |
| *k* | 0.300 | 0.270 | 0.060 | 0.0854 | 0.0777 | 0.0047 | 0.0347 |
| *l* | 0.150 | 0.150 | 0.015 | 0.0475 | 0.0432 | 0.0006 | 0.0006 |
| *m* | 0.100 | 0.180 | -0.070 | 0 | 0 | 0 | -0.0800 |
| *n* | 0.100 | 0.120 | -0.010 | 0 | 0 | 0 | -0.0200 |
| Total | 1.000 | 1.000 | | | | 0.0138 | 0.0138 |

Note: The subscripts ($j$, $k$, $l$, $m$, $n$) are suppressed; and $q = p_1 / p_0$ and $Q = P_1 / P_0$.

For convenience, we use the symbol $d^*$ (0, 1*, 2*, …, 5*) to represent the $d^*$th stage. At the $d^*$th stage, we use three strategies:

- if $\varepsilon_i a_{ie} - b_{ie} > 0$, then $v_i > 0$,
- if $\varepsilon_i a_{ie} - b_{ie} < 0$, then $v_i < 0$,
- if $\varepsilon_i a_{ie} - b_{ie} = 0$, then $v_i = 0$,

where the subscript $e$ represents the stage and $e = d^* - 1$. (The conventional subtraction is adapted for two cases below. For example, $3^* - 1 = 2^*$ and $1^* - 1 = 0 \equiv$ the initial stage.)

[4] Case 4: $\varepsilon_i = 1, v_i = x, \text{and} -0.05 \leq x \leq 0.05$ for all the $i$th firms.

This is the same as Case 1. We shall explain the method with Table 4 that shows the result.

At the first stage, we computed the $Cs$ for each firm. The $Cs$ column shows the value for each firm computed by (29). The $Cs$ values at the $d^*$th stage are always shown in these columns at the $d^* - 1$ stage. Because the $m$th firm had the largest $Cs = (a_{m0} - b_{m0})x = 0.0040$, we made the $m$th firm decrease its input by 5%. After the execution, we computed the new values of $y_{i1*}$, $h_{i1*}$, $a_{i1*}$, $b_{i1*}$, and so on. Some of these are exhibited in each column at the first stage. Note that the value of the $m$th firm exposed in column "$Cs$" at the initial stage nearly equals $(P_{1*}/P_0) - 1 = (Y_{1*}/H_{1*})/(Y_0/H_0) - 1$. The situation is similar for other stages.

Using the first stage result, we select the firm with the largest value of $Cs = (a_{i1*} - b_{i1*})x$, except for the $m$th firm, at the second stage. This was the $j$th firm's $Cs = (a_{j1*} - b_{j1*})x \approx 0.0035$ and we then made the $j$th firm increase its input by 5%. We iterated these procedures, which followed the order $m \to j \to k \to n \to l$. The final result is shown at the fifth stage. The values, $y_{i5*}$, $h_{i5*}$, $Y_{5*}$, and $H_{5*}$, at this stage correspond to those, $y_{i1}$, $h_{i1}$, $Y_1$, and $H_1$, derived by the single-stage method.

From the initial values and the results at the fifth stage, we will obtain the strategy $v_i = h_{i5*} / h_{i0} - 1$. The TLPR was about 1.00% up. (That of Case 1 is about 0.99% up.) At the fifth stage, we had $a_{l4*} - b_{l4*} = -0.0014$ and then made the $l$th firm decrease its input by 5%. Compare this regulation with that of Case 1.

Table 4 Regulated results of Case 4

| Firm | Initial stage | | | 1st stage | | | 2nd stage | | |
|---|---|---|---|---|---|---|---|---|---|
| | *y* (*Y*) | *h* (*H*) | *Cs* | *y* (*Y*) | *h* (*H*) | *Cs* | *y* (*Y*) | *h* (*H*) | *Cs* |
| *j* | 42.00 | 28.00 | 0.00350 | 42.00 | 28.00 | **0.00346** | **44.10** | **29.40** | |
| *k* | 36.00 | 27.00 | 0.00150 | 36.00 | 27.00 | 0.00145 | 36.00 | 27.00 | **0.00138** |
| *l* | 18.00 | 15.00 | 0.00000 | 18.00 | 15.00 | 0.00003 | 18.00 | 15.00 | 0.00006 |
| *m* | 12.00 | 18.00 | **0.00400** | **11.40** | **17.10** | | 11.40 | 17.10 | |
| *n* | 12.00 | 12.00 | 0.00100 | 12.00 | 12.00 | 0.00103 | 12.00 | 12.00 | 0.00103 |
| Total | 120.00 | 100.00 | | 119.40 | 99.10 | | 121.50 | 100.50 | |

| Firm | 3rd stage | | | 4th stage | | | 5th stage | | |
|---|---|---|---|---|---|---|---|---|---|
| | *y* (*Y*) | *h* (*H*) | *Cs* | *y* (*Y*) | *h* (*H*) | *Cs* | *y* (*Y*) | *h* (*H*) | *q* (*Q*) |
| *j* | 44.10 | 29.40 | | 44.10 | 29.40 | | 44.10 | 29.40 | 1 |
| *k* | **37.80** | **28.35** | | 37.80 | 28.35 | | 37.80 | 28.35 | 1 |
| *l* | 18.00 | 15.00 | 0.00006 | 18.00 | 15.00 | **0.00007** | **17.10** | **14.25** | 1 |
| *m* | 11.40 | 17.10 | | 11.40 | 17.10 | | 11.40 | 17.10 | 1 |
| *n* | 12.00 | 12.00 | **0.00102** | **11.40** | **11.40** | | 11.40 | 11.40 | 1 |
| Total | 123.30 | 101.85 | | 122.70 | 101.25 | | 121.80 | 100.50 | 1.00995 |

Note: The subscripts are suppressed; and $q = p_{5*} / p_0$ and $Q = P_{5*} / P_0$.

[5] Case 5: $\varepsilon_i = 1, v_i = x,$ and $-0.05 \le x \le 0.05$ for all the $i$th firms, and $H_1 = H_0$.

In practice, we may have to consider such a constraint in a certain industry in which total labor inputs are set by $H_1 = H_0$.We add this constrain to the conditions in Case 1. Because the decreased inputs are crucial, this method may be the best means of yielding the solution.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

Table 5 Regulated results of Case 5

| Firm | Initial stage | | | 1st stage | | | 2nd stage | | |
|---|---|---|---|---|---|---|---|---|---|
| | $y$ ($Y$) | $h$ ($H$) | $Cs$ | $y$ ($Y$) | $h$ ($H$) | $Cs$ | $y$ ($Y$) | $h$ ($H$) | $Cs$ |
| $j$ | 42.00 | 28.00 | | 42.00 | 28.00 | | 42.00 | 28.00 | |
| $k$ | 36.00 | 27.00 | | 36.00 | 27.00 | | 36.00 | 27.00 | |
| $l$ | 18.00 | 15.00 | | 18.00 | 15.00 | 0.00003 | 18.00 | 15.00 | **0.00004** |
| $m$ | 12.00 | 18.00 | **0.00400** | **11.40** | **17.10** | | 11.40 | 17.10 | |
| $n$ | 12.00 | 12.00 | 0.00100 | 12.00 | 12.00 | **0.00103** | **11.40** | **11.40** | |
| Total | 120.00 | 100.00 | | 119.40 | 99.10 | | 118.80 | 98.50 | |
| Firm | 3rd stage | | | 4th stage | | | 5th stage | | |
| | $y$ ($Y$) | $h$ ($H$) | $Cs$ | $y$ ($Y$) | $h$ ($H$) | $Cs$ | $y$ ($Y$) | $h$ ($H$) | $q$ ($Q$) |
| $j$ | 42.00 | 28.00 | **0.00349** | **44.10** | **29.40** | | 44.10 | 29.40 | 1 |
| $k$ | 36.00 | 27.00 | 0.00146 | 36.00 | 27.00 | **0.00087** | **37.13** | **27.85** | 1 |
| $l$ | **17.10** | **14.25** | | 17.10 | 14.25 | | 17.10 | 14.25 | 1 |
| $m$ | 11.40 | 17.10 | | 11.40 | 17.10 | | 11.40 | 17.10 | 1 |
| $n$ | 11.40 | 11.40 | | 11.40 | 11.40 | | 11.40 | 11.40 | 1 |
| Total | 117.90 | 97.75 | | 120.00 | 99.15 | | 121.13 | 100.00 | 1.00944 |

Note: The subscripts are suppressed; and $q = p_{5*} / p_0$ and $Q = P_{5*} / P_0$.

Our method comprises of two parts, namely, the decreasing input and the increasing input. We explain the method using Table 5 that shows the result.

In the first part, we select the $i$th firm that satisfies $a_{ie} - b_{ie} < 0$ and has the largest $Cs$ at the $d$*th stage, and make this firm decrease its input by 5% as in Case 4. The order was $m \rightarrow n \rightarrow l$. We also computed the decreased-input sum $h^* = 0.05(h_{m0} + h_{n1*} + h_{l2*}) = 2.25$.

In the second part, we allocate the sum to the other firms (the $j$th and $k$th firms). Our procedures are as follows. At the fourth stage, we select a firm that satisfies $a_{i3*} - b_{i3*} > 0$ and has the largest $Cs$. This was the $j$th firm. So, we computed $h^* / h_{j3*} = \omega_{4*} \approx 0.080 > 0.05$. (If $\omega_{4*} \leq 0.05$, we will make the $j$th firm increase its input by $100\omega_{4*}$ percent and finish the allocation.) Thus we made the $j$th firm increase its input by 5% and computed the residual $h^{**} = h^* - 0.05h_{j3*} = 0.85 > 0$. At the fifth stage, we had $a_{k4*} - b_{k4*} > 0$ and computed $h^{**} / h_{k4*} = \omega_{5*} \approx 0.0315 < 0.05$. Hence we made the $k$th firm increase its input by $100\omega_{5*}$ percent.

From the initial values and the results at the fifth stage, we will obtain the strategy $v_i = h_{i5*} / h_{i0} - 1$. While the $m$th, $n$th and $l$th firms decreased their inputs by 5%, the $j$th and $k$th firms increased their inputs, respectively, by 5% and about 3.15%. Note that $H_0 \approx H_{5*}$. The TLPR was increased by about 0.94%.

## 8. Concluding remarks

In this study, we have derived the single-handed and double-handed aggregative formulae for labor productivity quotient at the firm-level. Using one of these formulae, we have focused on the strategic decomposition of TLPR. Assuming a planned economy, we considered a situation wherein a government had to regulate TLPR in a certain industry. Our problem then becomes how to increase TLPR by as much as possible during a few years, a problem that can be solved by the strategic decomposition. We have demonstrated how to apply our strategic decomposition to the problem using numerical examples.

This paper shows many aggregative formulae, some of which are systematically derived herein. These formulae derive different values for each firm's contribution to TLPQ (or TLPR). Relevant questions are whether one aggregative formula is superior to the others, and the meaning of the aggregative formulae. Based on the above discussions and the results shown in Appendix C, the formulae are unlikely to have practical meanings. Therefore we want to leave this matter open.

Subjects such as our strategic decomposition and its applications have never been examined, and so we make some remarks on them as follows.

1) How to increase TLPR

To solve this problem, we have proposed single-stage and multi-stage methods. Our strategic decomposition (27) can be used with both of these methods.

2) *Ex ant* information against *ex post* information

To regulate TLPR, we must use *ex ant* information such as two shares at the base period ($a_{i0}$ and $b_{i0}$) and also use the two control variables ($u_i$ and $v_i$). If all the firms' u-v ratios $\varepsilon_i$ defined by (19) are *ex ant* information, our control variables turn out an $N$-vector of the firms' input changes ($N$ is the number of firms). To regulate TLPR, *ex post* information such as two shares at the comparison period ($a_{i1}$ and $b_{i1}$) is not helpful. Remember that $a_{i1}$ is dependent on all firms' $u_j$ and $b_{i1}$ is dependent on all firms' $v_j$. Notably, the single-handed additive aggregations ((5) and (6)) and all the multiplicative aggregations ((13), (14), (15), (16), and so on) demand *ex post* information as these arguments. See also (32) and (34) in Appendix A.

3) Most important signal

If all $\varepsilon_i$ are *ex ant* information as above, the most important signal to regulate TLPR is $\varepsilon_i a_{i0} - b_{i0}$ (the difference between the $i$th firm's weighted output share at the base period and its input share). This material fact is easily understood from the discussions in Section 7. While we repeatedly demonstrated major roles for $a_{i0}$ and $b_{i0}$, $\varepsilon_i$ is also a key player in the regulation of TLPR, being related to TFP and capital.

4) Misleading signal

To increase TLPQ (or TLPR), firm productivity at the base period $p_{i0}$ and the quotients $p_{i1}/p_{i0}$ are misleading signals. It is false to say that reallocation of labor from the lower-productivity firm to the higher-productivity firm will produce total labor productivity

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

growth (i.e., $P_1/P_0 > 1$)[17] and that productivity growth at the firm-level (i.e., $p_{i1}/p_{i0} > 1$) will lead to productivity growth at the industry-level.[18]

5) Requisite information

The data needed for the regulation of TLPR are firms' (or industries') two shares ($a_{i0}$ and $b_{i0}$) and u-v ratios ($\varepsilon_i$) that can be used as *ex ante* information; and their control variables ($v_i$) whose values or ranges are determined by something as in Subsections 7.2 and 7.3. Presently we can access some databases at the industry-level, such as EU KLEMS and OECD STAN.[19] The former provides annual measures of outputs and inputs at the industry-level for many countries, from which we can produce the two shares at the base period[20] and u-v ratios. Examining these data for Japan and the United States, we found the annual variations of outputs and inputs to be relatively small (so (22) holds at the industry-level for developed countries such as Japan). However, we found the u-v ratios to have very large annual variations, and thus the u-v ratios at the industry-level are not used as *ex ant* information.[21] Estimating the u-v ratio at a firm-level may be easy and that at an industry-level may not; and they are beyond the scope of this paper. For these topics including the database, see CompNet task force (2014).

[17] Dynamic allocative efficiency by Haltiwanger (2011) is this reallocation of resources including labor.
[18] See the essential strategies and Table 1 in Section 7.
[19] Available at http://www.euklems.net, and http://www.oecd-ilibrary.org.
[20] Naturally, we can use these shares as *ex ante* information.
[21] These results are available from the author upon request.

**References**

Balk, B. M. (2008) *Price and quantity index numbers*: *Models for measuring aggregate change and difference*, Cambridge University Press, New York.

Balk, B. M. (2014) “Dissecting aggregate output and labour productivity change,” *Journal of Productivity Analysis*, 42: 35–43.

Berthou, A. and Standoz, C. (2014) “Labour productivity in Europe: Allocative efficiency of labour or performance of firms?” *Quarterly Selection of Articles* (Banque de France), no. 34: 47–67.

BLS (US Bureau of Labor Statistics) (2008) *Technical Information about the BLS major sector productivity and costs measures*.

http://www.bls.gov/lpc/lpcmethods.pdf (accessed May 20, 2015).

Carlson, B. C. (1972) “The logarithmic means,” *American Mathematical Monthly*, 79(6): 615–618.

CompNet Task Force (2014) “Micro-based evidence of EU competitiveness: The CompNet database,” *ECB* (European Central Bank) *Working Paper*, no. 1634.

De Avillez, R. (2012) “Sectoral contributions to labor productivity growth in Canada: Does the choice of decomposition formula matter?” *International Productivity Monitor*, 24: 97–117.

Dean, E. R. (1999) “The accuracy of the BLS productivity measures,” *Monthly Labor Review*, 122(2): 24–34.

Diewert, W. E. (2015) “Decompositions of productivity growth into sectoral effects,” *Journal of Productivity Analysis*, 43: 367–387.

Dumagan, J. C. (2013) “A generalized exactly additive decomposition of aggregate labor productivity growth,” *Review of Income and Wealth*, 59: 157–168.

European Commission (2004) *European economy* no. 6 (The EU economy: 2003 review), European Communities.

Fox, K. J. (2012) “Problems with (dis)aggregating productivity, and another productivity paradox,” *Journal of Productivity Analysis*, 37: 249–259.

Haltiwanger, J. (2011) “Firm dynamics and productivity growth,” *EIB* (European Investment Bank) *Papers*, 16(1): 116–136.

Holman, C., Joyeux, B., and Kask, C. (2008) “Labor productivity trends since 2000, by sector and industry,” *Monthly Labor Review,* 131(2): 64–82.

ILO (International Labour Organization) (2014) *Key indicators of the labour market*, 8th edn. Available on ILO website (http://www.ilo.org), (accessed May 20, 2015).

Jorgenson, D. W. and Stiroh, K. J. (2001) “Raising the speed limit: U.S. economic growth in the information age,” In: Jorgenson, D. W. and Lee, F. C. (eds) *Industry-level*

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

*productivity and international competitiveness between Canada and the United States*, Industry Canada, Ottawa, pp 5–75.

Lewrick, U., Mohler, L., and Weder, R. (2014) “When firms and industries matter: Understanding the sources of productivity growth,” *BIS* (Bank for International Settlements) *Working Papers*, no. 469. Available on BIS website (http://www. bis.org), (accessed May 20, 2015).

Nordhaus, W. D. (2002) “Productivity growth and new economy,” *Brookings Papers on Economic Activity*, issue 2: 211–265.

OECD (2001) *Measuring productivity: Measuring of aggregate and industry-level productivity growth*, OECD, Paris.

OECD (2014) *Perspectives on global development 2014: Boosting productivity to meet the middle-income challenge*, OECD Publishing, Paris.

OECD (2015) *OECD compendium of productivity indicators 2015*, OECD Publishing, Paris.

O’Mahony, M. and Timmer, M. P. (2009) “Output, input and productivity measures at the industry level: The EU KLEMS database,” *Economic Journal*, 119: 374–403.

Pilat, D. and Schreyer, P. (2004) “The OECD productivity database: An overview,” *International Productivity Monitor*, no. 8: 59–65.

Reinsdorf, M. (2015) “Measuring industry contributions to labour productivity change: A new formula in a chained Fisher Index framework,” *International Productivity Monitor*, 28, 3–26.

Schreyer, P. and Pilat, D. (2001) “Measuring productivity,” *OECD Economic Studies*, no. 33: 127–170.

Stiroh, K. (2002) “Information technology and the U.S. productivity revival: What do the industry data say?” *American Economic Review*, 92(5): 1559–1576.

Stolarsky, K. B. (1975) “Generalizations of the logarithmic mean,” *Mathematics Magazine*, 48(2): 87–92.

Syverson, C. (2011) “What determines productivity?” *Journal of Economic Literature*, 49 (2): 326–365.

Tang, J. and Wang, W. (2004) “Sources of aggregate labour productivity growth in Canada and the United States,” *Canadian Journal of Economics*, 37: 421–444.

Timmer, M. P., Inklaar, R., O’Mahony, M., and Ark, B. (2010) *Economic growth in Europe*, Cambridge University Press, New York.

Timmer, M. P., O’Mahony, M., and Ark, B. (2007) “EU KLEMS growth and productivity accounts: An overview,” *International Productivity Monitor*, no.14: 71–85.

Tsuchida, S. (1997) “A family of almost ideal log-change index numbers,” *Japanese Economic Review*, 48: 324–342.

Tsuchida, S. (2014) "Interconvertible rules between an aggregative index and a log-change index," *Electronic Journal of Applied Statistical Analysis*, 7 (2): 394–415.

Vartia, Y .O. (1976) "Ideal log-change index numbers," *Scandinavian Journal of Statistics*, 3: 121–126.

Vijselaar, F. and Albers, R. (2004) "New technologies and productivity growth in the Euro area," *Empirical Economics*, 29: 621–646.

*International Journal of Contemporary Economics and Administrative Sciences*
*ISSN: 1925 – 4423*
*Volume :5, Issue:3-4, Year:2015, pp.109 – 138*

**Appendix A: Other formulae of total labor productivity quotient**

We shall briefly discuss other formulae of TLPQ not covered in Sections 2 and 3. As above, we use two fundamental relations derived from $a_{it}/b_{it} = p_{it}/P_t$:

$$\frac{P_1}{P_0} = \frac{a_{i0}b_{i1}p_{i1}}{a_{i1}b_{i0}p_{i0}}. \tag{30}$$

$$\begin{aligned}\Delta\log P &= \Delta\log p_i - \Delta\log a_i + \Delta\log b_i \\ &= \Delta\log p_i - \Delta\log(a_i/b_i) = \Delta\log p_i + \Delta\log(b_i/a_i)\end{aligned} \tag{31}$$

Multiplying both sides in (30) by one of the shares (or these ratios) and summing over all the firms, we obtain a single-handed (or double-handed) additive formula. Similarly, multiplying both sides in (31) by a logarithmic mean of one of the shares (or these ratios) and summing over all the firms yields a single-handed (or double-handed) multiplicative formula.

First, we show the single-handed formula. Multiplying both sides in (30) by $a_{i1}$ and summing over all the firms, we obtain (5). Replacing $a_{i1}$ with $b_{i0}$, we can derive (6). Multiplying both sides in (31) by $L(a_i)$ and summing over all the firms yields (13). Replacing $L(a_i)$ with $L(b_i)$, we can derive the other multiplicative formula shown in Subsection 3.3.

Next, we show the double-handed formulae. Multiplying both sides in (30) by $a_{i1}/b_{i1}$ and summing over all the firms yields:

$$\frac{P_1}{P_0} = \sum{}' \left(\frac{a_{i0}/b_{i0}}{\Sigma(a_{j1}/b_{j1})}\right)\left(\frac{p_{i1}}{p_{i0}}\right) = \frac{P_1}{P_0}\sum{}' \frac{p_{i1}}{(\Sigma p_{j1})}. \tag{32}$$

Similarly, multiplying both sides in the above by $b_{i0}/a_{i0}$ and summing over all the firms, we obtain another double-handed additive formula:

$$\frac{P_1}{P_0} = \sum{}' \left(\frac{b_{i1}/a_{i1}}{\Sigma(b_{j0}/a_{j0})}\right)\left(\frac{p_{i1}}{p_{i0}}\right) = \frac{P_1}{P_0}\sum{}' \frac{1/p_{i0}}{(\Sigma\, 1/p_{j0})}. \tag{33}$$

Comparing (33) with (32), we see that the former causes the IN-C and the latter does not (see the next appendix).

Multiplying both sides in (31) by $L(a_i/b_i)$ and summing over all the firms, we find the multiplicative formula:

$$\Delta\log P = \frac{1}{\Sigma L(a_j/b_j)}\Sigma\left(L(a_i/b_i)\Delta\log p_i - \frac{a_{i1}}{b_{i1}} + \frac{a_{i0}}{b_{i0}}\right),$$

or

$$\Delta \log P = \sum{}' \lambda(a_i/b_i)\left(\Delta \log p_i - \Delta \log(a_i/b_i)\right) \\ = \Delta \log P \left(\sum{}' \lambda(a_i/b_i)\right), \tag{34}$$

wherein we used the logarithmic mean:

$$L(a_i/b_i) = \frac{\Delta(a_i/b_i)}{\Delta \log(a_i/b_i)} = \frac{(a_{i1}/b_{i1}) - (a_{i0}/b_{i0})}{\Delta \log a_i - \Delta \log b_i}$$

and $\lambda(a_i/b_i) = L(a_i/b_i)/\Sigma L(a_j/b_j)$. To enable the *i*th firm to contribute to the logarithmic TLPQ without doing anything, we have only to assume $P_1 \neq P_0$.[22] Compare this property with those of (15), (16), and (17). If the logarithmic mean can be approximated by the geometric mean $G(x)$, that is, $L(x) \approx G(x) \equiv (x_1 x_0)^{0.5}$, we have

$$\lambda(a_i/b_i) \approx G(p_i/P)/\left(\Sigma G(p_j/P)\right) = G(p_i)/\left(\Sigma G(p_j)\right).$$

Substituting this approximation into (34), we can see that (34) corresponds to (32).

Replacing $L(a_i/b_i)$ with $L(b_i/a_i)$, we have:.

$$\Delta \log P = \frac{1}{\Sigma L(b_j/a_j)} \sum{}' \left( L(b_i/a_i) \Delta \log p_i + \frac{b_{i1}}{a_{i1}} - \frac{b_{i0}}{a_{i0}} \right) \\ = \Delta \log P \left(\sum{}' \lambda(b_i/a_i)\right),$$

wherein $L(b_i/a_i)$ is defined similarly to the above and $\lambda(b_i/a_i)$

$= L(b_i/a_i)/\Sigma L(b_j/a_j)$. While this corresponds to (33), it does not cause the IN-C (see also the next appendix).

Ingenious procedures can convert (34) into a novel form. We obtain the following relations from the logarithmic means $L(P)$ and $L(p_i)$:[23]

$$\left(\frac{P_1}{P_0}\right)^{1/\Delta P} = \exp\left(\frac{1}{L(P)}\right), \left(\frac{p_{i1}}{p_{i0}}\right)^{1/\Delta p_i} = \exp\left(\frac{1}{L(p_i)}\right),$$

$$\lim_{\Delta P \to 0} \left(\frac{P_1}{P_0}\right)^{1/\Delta P} = \exp\left(\frac{1}{P_0}\right) = \exp\left(\frac{1}{P_1}\right), \lim_{\Delta p_i \to 0} \left(\frac{p_{i1}}{p_{i0}}\right)^{1/\Delta p_i} = \exp\left(\frac{1}{p_{i0}}\right) \\ = \exp\left(\frac{1}{p_{i1}}\right).$$

These relations and (34) produce the novel form given by

[22] If $p_{i1} = p_{i0}$ and $P_1 \neq P_0$, $(a_{i1}/b_{i1}) \neq (a_{i0}/b_{i0})$.
[23] Taking the logarithm of both sides in the first relation, we can find the definition of $L(P)$. The same can be done for other relations.

International Journal of Contemporary Economics and Administrative Sciences
ISSN: 1925 – 4423
Volume :5, Issue:3-4, Year:2015, pp.109 – 138

$$\frac{\Delta \log P}{\Delta P} = \sum{}' \lambda(a_i/b_i) \frac{L(p_i)}{L(P)} \frac{\Delta \log p_i}{\Delta p_i}.$$

$$\therefore \left\{ \left( \frac{P_1}{P_0} \right)^{1/\Delta P} \right\} = \prod \left[ \left\{ \left( \frac{p_{i1}}{p_{i0}} \right)^{1/\Delta p_i} \right\}^{\theta_i} \right]. \tag{35}$$

Here $\theta_i = \lambda(a_i/b_i)L(p_i)/L(P) > 0$. It is important to bear in mind that we must first calculate the term in the left (or right) braces if $\Delta P$ (or $\Delta p_i$) approaches zero.

If $P_1/P_0 \neq 1$ (i.e., $\Delta P \neq 0$) and $p_{i1}/p_{i0} \neq 1$ (i.e., $\Delta p_i \neq 0$) for all $i$, we obtain the TLPQ as follows:

$$\frac{P_1}{P_0} = \prod \left( \frac{p_{i1}}{p_{i0}} \right)^{\varphi_i},$$

which is the weighted geometric mean of the firm's productivity quotient $p_{i1} / p_{i0}$; and the weight is $\varphi_i = \lambda(a_i/b_i)\Delta \log P/(\Delta \log p_i)$ and may be negative.

## Appendix B: Invariable contribution to total labor productivity growth

We shall demonstrate that some aggregations connote a firm's invariable contribution to TLPQ, logarithmic TLPQ, or TLPR. This defect (i.e., IN-C) occurs in all single-handed aggregations and some double-handed additive aggregations. We can find the defect if each firm's contribution is given by:

$$\frac{P_1}{P_0} = \frac{P_1}{P_0} \sum{}' x_i, \ \frac{\Delta P}{P_0} = \frac{\Delta P}{P_0} \sum{}' x_i, \text{ or } \Delta \log P = \Delta \log P \sum{}' x_i$$

where $x_i$ is independent of one or both of the two control variables $u_i$ and $v_i$ while $\Sigma x_i = 1$. Usually, $x_i$ is decomposed into some factors (see, for example, Reinsdorf 2015). We can also produce variants, such as

$$\frac{\Delta P}{P_0} = \frac{P_1 \Sigma x_i - P_0 \Sigma z_i}{P_0},$$

where $z_i$ is independent of the two control variables, $P_0 z_i$ is a predetermined variable, and $\Sigma z_i = 1$.

As an additive form, we take (5), (33), and (20). From the extreme right of (5), we can see that this formula cannot mirror changes in firm inputs. Consider the example where output changes do not occur for all firms despite changes in their inputs. Herein, $u_i = 0$, $v_i > 0$, and $a_{i1} = a_{i0}$ for all $i$. Thus, the TLPQ is

$$\frac{P_1}{P_0} = \frac{1}{1 + \Sigma b_{i0} v_i}.$$

There are many vectors **v** = $\{v_i\}$ that meet $\Sigma b_{i0} v_i = d$, where $d$ is any constant. Provided that the firms' input changes are any one of these vectors, $P_1/P_0$ remains constant and thus each firm's contribution as derived by (5) is also constant. (Note that $a_{i1}$ in (5) is independent of $v_i$ and $b_{i0}$ in (6) is independent of $u_i$ and $v_i$.)

Additionally, many sets of two vectors **u** = $\{u_i\}$ and **v** = $\{v_i\}$ produce the same $P_1/P_0$, and each $p_{i0}$ is independent of the two control variables. Thus, it is evident that (33) causes the IN-C. (Compare this property with that derived by (32). Recall that $p_{i1}$ is dependent of the two control variables.)

The familiar decomposition (20) causes a defect similar to (5), since the awkward expansion on its right hand side leads to

$$\frac{\Delta P}{P_0} = \sum{}' \left( \frac{a_{i0}(p_{i1} - p_{i0})}{p_{i0}} + \frac{u_{i0} p_{i1}(b_{i1} - b_{i0})}{b_{i0} p_{i0}} \right)$$

$$= \sum{}' \left( \frac{1}{Y_0} \left( \frac{y_{i1} h_{i0}}{h_{i1}} - y_{i0} \right) + \frac{H_0}{Y_0} \left( \frac{y_{i1}}{H_1} - \frac{y_{i1} h_{i0}}{H_0 h_{i1}} \right) \right)$$

$$= \frac{1}{P_0} \left( \sum{}' P_1 u_{i1} - \sum{}' P_0 u_{i0} \right)$$

wherein we used the relations such as $u_{i0} = y_{i0}/Y_0$, $p_{it} = y_{it}/h_{it}$, and so on ($t$ = 0 and 1). Naturally, the final result is directly obtained from the following numerator of (20):

$$\Delta P = P_1 - P_0 = \Sigma b_{i1} p_{i1} - \Sigma b_{i0} p_{i0} = \Sigma P_1 u_{i1} - \Sigma P_0 u_{i0}.$$

From the example in (5) above, we have $a_{i1} = a_{i0}$ for all $i$. Thus we have

$$\frac{\Delta P}{P_0} = \frac{\Delta P}{P_0} \sum{}' u_{i0}.$$

As a multiplicative form, we take (13) and (14). From (13) and (31), we have

$$\Delta \log P = \sum \lambda(a_i)(\Delta \log p_i + \Delta \log b_i)$$

$$= \sum{}' \lambda(a_i)(\Delta \log P + \Delta \log u_i)$$

$$= \Delta \log P \sum{}' \lambda(a_i)$$

because $\Sigma L(a_i) \Delta \log u_i = 0$. (Compare the final formula with (34), which is the double-handed formula). Thus, firms' contributions to the logarithmic TLPQ derived by (13) cannot reflect changes in their inputs, since $\lambda(a_i)$ are independent of all the control variables $v_j$.

Similarly, (14) leads to

$$\Delta \log P \approx \sum{}' A(a_i)(\Delta \log P + \Delta \log u_i) \approx \Delta \log P \sum{}' A(a_i),$$

*International Journal of Contemporary Economics and Administrative Sciences*
*ISSN: 1925 – 4423*
*Volume :5, Issue:3-4, Year:2015, pp.109 – 138*

wherein we used $\Sigma A(a_i)\Delta \log u_i \approx \Sigma \Delta a_i = 0$ (note that $A(a_i) \approx L(a_i) = \Delta a_i / \Delta \log u_i$).

Table 6 Each firm's contribution to TLPR

| Firm | Equation | Approximations | | |
|---|---|---|---|---|
| | (20) | (21) | (23) | (25) |
| *j* | 0.01441 | 0.07875 | -0.06136 | 0.07845 |
| *k* | 0.01235 | 0.03563 | -0.02353 | 0.03490 |
| *l* | 0.00617 | 0.00187 | 0.00276 | 0.00128 |
| *m* | -0.01041 | -0.07625 | 0.07932 | -0.07409 |
| *n* | -0.00315 | -0.02000 | 0.02121 | -0.02061 |
| Total | 0.01937 | 0.02000 | 0.01840 | 0.01994 |

**Appendix C: Firms' contributions to TLPR**

We shall exemplify the firms' contributions to the TLPR derived by (20), (21), (23), and (25). Table 6 exhibits these values using the results shown in Table 2. (The values derived by (21) have already been shown in Table 2.) The values of firms' contributions vary in accordance with the formulae. Additionally, all the formulae have negative values. The meaning of these negative values is an important question, particularly given that our essential strategies always produce non-negative strategic components.